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Abstract

We introduce the notion of completely 2-absorbing (denoted by, c-2-absorbing) ideal of an N-group G, as a generalization of completely prime ideal of module over a right near-ring N. We obtain that, for an ideal I of a monogenic N-group G, if $(I: G)$ is a c-2-absorbing ideal of N, then I is a c-2-absorbing ideal of G. The converse also holds only when G is locally monogenic over a distributive near-ring N. We discuss the properties such as homomorphic images, inverse images of c-2-absorbing ideals of G. Examples of c-2-absorbing ideals of N-groups are given where N is non-commutative and in the sequel some results of 2-absorbing ideals from module over rings are generalized to N-groups.

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Keywords: *Near-ring, Completely prime ideal, IFP, monogenic, N-group.*

1. Introduction

A natural generalization of module over a ring is a module over an arbitrary near-ring. Precisely it is an action of a group (not necessarily abelian) over a near-ring. Juglal et.al.[16] generalized prime ideal in modules over rings to modules over near-rings in several ways. They obtained some characterizations of prime near-ring modules and exhibited interesting properties. More importantly, connection between a prime ideal of a module over a near-ring and the corresponding annihilators in a near-ring were investigated. Badawi [1] introduced the idea of 2-absorbing ideal of a commutative ring with identity, as a generalization of prime ideal in a commutative ring and explored its properties. Darani and Soheilnia [6] introduced 2-absorbing submodule of module over commutative ring with identity; later studied by Payrovi and Babaei [24],

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while generalizations of 2-absorbing modules over non-commutative rings were given by Groenewald and Nguyen [10]. Prime submodule of module over ring generalizes the notion of prime ideal of rings. In [8, 9], the author extensively studied prime submodules of modules over rings. Indeed, there are several means to generalize these to prime ideals of near-rings. Prime ideals in near-rings have been introduced by Holcombe [15], and later studied by [3, 5, 18, 19]. Due to lack of one distributive property, and addition operation in general, not abelian, the authors were able to study various types of prime ideals. Recently, Hamsa et.al. [12] defined quasi associatively in $\Theta\Gamma$-N-groups (generalized N-groups), and proved the fundamental isomorphism theorems. In [11], the author introduced the notion weakly prime radical of an ideal and initiated the study of weakly completely prime ideals, further, investigated such rings in which every proper ideal is weakly completely prime. In present paper, we define the notions c-2-absorbing ideals of near-ring and 2-absorbing ideals of a module over a near-ring, which generalize the notions completely prime ideal of a near-ring and completely prime ideal of a module over a near-ring respectively. We organize the paper into four sections, in section 2, we provide essential preliminary notions required in this paper. In section 3, we introduce the concept of completely 2-absorbing ideal of an N-group and prove some of its properties with suitable illustrations. In section 4 we consider completely 2-absorbing ideal of an N-group where N is a zero-symmetric near-ring and obtain subsequent results. Throughout, we use N to denote a right near-ring not necessarily zero symmetric, and in section 4, we assume N to be zero-symmetric, and G stands for an N-group. Further, all other undefined notions and conventions will be as in Pilz [25].

2. Preliminaries

A (right) near-ring $(N, +,)$ is an algebraic system, in which N is an additive group (need not be abelian) and a multiplicative semi-group, satisfying only one distributive axiom: $(x + y)c = xc + yc$ for all $x, y, c \in N$. The definition referred as a right near-ring because it satisfies the right distributive axiom. Note that $0 \cdot m = 0$ and $(-m)n = -mn$, for all $m, n \in N$, but in general, $a0 \neq 0$ for some $a \in N$. If $m0 = 0$, for all $m \in N$, then N is zero-symmetric, we denote as $N = N_0$. For a group $(G, +)$, (not necessarily abelian), the set $M(G)$ of all maps h: $G \to G$ determines the structure of a near-ring, which is not a ring, under point-wise addition and function substitution. Obviously, for some appropriate group $(G, +)$, every near-ring can be embedded in $M(G)$. A subgroup $(A, +)$ of $(N, +)$ is said to be an ideal if it satisfies: (i) $(A, +)$ is normal, (ii) $AN \subseteq I$, (iii) $n(n_1 + a) - nn_1 \in A$ for all $n, n_1 \in N$, $a \in A$. Let $(G, +)$ be a group with 0_G, as additive identity. G is said to be an N-group (denoted by $_NG$ or simply G) if there corresponds a map $N \times G \rightarrow G$ (the image of (x, g) is written as xg), satisfy (i) $(x + y)g = xg + yg$, and (ii) $(xy)g = x(yg)$ for all $g \in G$ and $x, y \in N$. It is evident that every near-ring is an N-group N (over itself). Also, each module over a ring N (elements are written on the left) is an N-group. A subgroup $(A, +)$ of $_NG$ with $NA \subseteq A$ is called a N-subgroup of G. A normal subgroup A of G is called ideal of $_NG$ if $x(g + a) - xg \in A$ for all $x \in N$, $a \in A$, $g \in G$. For any two N-groups G_1 and G_2, a map h: $G_1 \rightarrow G_2$ is called an N-homomorphism, if for all $a, b \in G_1$ and $n \in N$, $h(a + b) = h(a) + h(b)$ and $h(na) = nh(a)$ hold. We call h an N-epimorphism if h is onto, and N-monomorphism if h is 1-1. It is evident that kernels of N-homomorphisms are the ideals of G. A near-ring N is distributive if $x(a + b) = xa + xb$ for all $x, a, b \in N$. As in [5], we use $< g >$ to denote the ideal generated by an element $g \in G$.

Definition 2.1 : (Pilz [25], Ramakotiah and Rao [26]) A proper ideal P of N is called prime if a for any two ideals S and T of N with $ST \subseteq P$ implies that $S \subseteq P$ or $T \subseteq P$; and P is completely prime (denoted as c-prime) if $st \in P$ implies $s \in P$ or $t \in P$. In case of commutative rings, the notions prime and c-prime coincide.

Definition 2.2 : (Bhavanari and Rao [2]) An ideal A of G is said to have insertion of factors property (denoted as, IFP) if $x \in N$, $g \in G$ with $xg \in A$ then $xng \in A$ for all $n \in N$. If (0_G) is an IFP ideal, then we call G as an IFP N-group.

3. Completely 2-absorbing ideals

The following definition generalizes the c-prime ideal of N.

Definition 3.1 : An ideal I of N is called completely 2-absorbing (abbr. c-2-absorbing) of N if whenever $x, y, z \in N$ with $xyz \in I$ then $xy \in I$ or $yz \in I$ or $xz \in I$. N is called c-2-absorbing if the ideal (0) is a c-2-absorbing ideal of N.

Definition 3.2 : (Juglal et.al. [16]) An ideal P of G with $(P \neq G)$ is called completely prime if $xg \in P$ imply $xG \subseteq P$ or $g \in P$ for all $x \in N$ and $g \in G$.

Definition 3.3 : A proper ideal P of G is called strictly completely prime if $xg \in P$ imply $x\langle g\rangle \subseteq P$ or $g \in P$.

Remark 3.4 : Every completely prime ideal is strictly completely prime and the converse need not be true, in general.

***Proof** :* Suppose P is c-prime. Let $x \in N$ and $g \in G$ such that $xg \in P$. Since P is c-prime, $xG \subseteq P$ or $g \in P$. Now $x\langle g\rangle \subseteq x\,G \subseteq P$ or $g \in P$. Therefore, P is strictly c-prime.

The converse need not be true, refer to the example 3.6 for strictly c-prime but not c-prime. We generalize the definition 3.2 as follows, which is a key notion in this paper. Examples are also given when N is not zero-symmetric.

Definition 3.5 : An ideal I of G is said to be a completely 2-absorbing (abbr. c-2-absorbing) ideal if whenever $x, y \in N$, $g \in G$ with $xyg \in I$ then $xy \in (I{:}G)$ or $xg \in I$ or $yg \in I$.

Example 3.6 : Take $G = Z$ and $N = Z$, near-ring of integers. Then G is an N-group. Clearly $I_1 = \{0, 4\}$ and $I_2 = \{0, 2, 4, 6\}$ are ideals of G. It can be verified that I_1 is a c-2-absorbing ideal of G but not c-prime, since 2.6 $\in I_1$ and neither $2Z_8 \subseteq I_1$ nor $6 \in I_1$. The ideal I_2 is c-prime as well as c-2-absorbing. However, I_1 is strictly c-prime, since $2\langle 6\rangle = \{0, 4\} \subseteq I_1$.

Example 3.7 : Refer the near-ring N, given in E-23, page 408 of Pilz [25], where $N = \{0, a, b, c\}$. Consider N itself as an N-group with respect to the operations + and · are given respectively in table 1 and table 2 below.

Table 1

+	0	*a*	*b*	*c*
0	0	*a*	*b*	*c*
a	*a*	0	*c*	*b*
b	*b*	*c*	0	*a*
c	*c*	*b*	*a*	0

Table 2

.	0	*a*	*b*	*c*
0	0	0	0	0
a	*a*	*a*	*a*	*a*
b	*b*	*b*	*b*	*b*
c	*c*	*c*	*c*	*c*

Then clearly, N is not zero-symmetric. It can be easily seen that, $\{0\}$ is c-prime and c-2-absorbing.

Example 3.8 : Take $G = Z_6$ and $N = Z$, near-ring of integers. Then G is an N-group. Clearly $I_1 = \{0\}$ and $I_2 = \{0, 2, 4\}$ are ideals of G. It can be verified that I_1 is a c-2-absorbing ideal of G, but not c-prime, since 2.3 $\in I_1$ and neither $2Z_6 \subseteq I_1$ nor $3 \in I_1$. The ideal I_2 is c-prime as well as c-2-absorbing.

Example 3.9 : Let $G = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$, where entries of matrices in G are from Z_2 and $N = M_2(Z)$. Then G is a N-group. Clearly, $I = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$ is an ideal of G, and is c-2-absorbing, but not c-prime, since $\begin{pmatrix} 3 & 5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} =$ $\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \in I$ but neither $\begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \in I$ nor $\begin{pmatrix} 3 & 5 \\ 1 & 7 \end{pmatrix} G \subseteq I$.

Example 3.10 : Let $N = (S_3, +, \cdot)$ be a near-ring (given in H-11, pg. 410 of Pilz [25]), which is not zero-symmetric and non-commutative. Consider N as an N-group over itself, where the operations + and · are given respectively in table 3 and table 4.

Table 3

+	0	*a*	*b*	*c*	*x*	*y*
0	0	*a*	*b*	*c*	*x*	*y*
a	*a*	0	*y*	*x*	*c*	*b*
b	*b*	*x*	0	*y*	*a*	*c*
c	*c*	*x*	*y*	0	*b*	*a*
x	*x*	*b*	*c*	*a*	*y*	0
y	*y*	*c*	*a*	*b*	0	*x*

Table 4

·	0	*a*	*b*	*c*	*x*	*y*
0	0	0	0	0	0	0
a	*a*	*a*	*a*	*a*	*a*	*a*
b	*a*	*a*	*c*	*b*	*b*	*c*
c	*a*	*a*	*b*	*c*	*c*	*b*
x	0	0	*y*	*x*	*x*	*y*
y	0	0	*x*	*y*	*y*	*x*

Clearly $I = \{0\}$ is c-2-absorbing, but not c-prime, since $xa \in$ I, and $a \notin$ I, $xG \not\subset I$.

Theorem 3.11 : *Let h be an N-homomorphism of N-groups G onto G^1. If A is a c-2-absorbing ideal of G containing ker h, then h(A) is a c-2-absorbing ideal of G′.*

Proof : Take a, $b \in N$, $g' \in G^1$ with $abg' \in h(A)$. Then $abg' = h(x)$ for some $x \in A$.

Since h is an N-epimorphism and $g' \in G'$, it follows that $h(g) = g'$ for some $g \in G$.

Since h is an N-homomorphism, $h(abbg) = abh(g) = abg' = h(x)$, and hence, $h(abg\text{-}x) = 0$ in G^1. Since $x \in A$ and $abg\text{-}x \in \ker h \subseteq A$, we have, $abg \in A$.

Since A is c-2-absorbing and $abg \in A$, we have,

$ab \in (A\!: G)$ or $ag \in A$ or $bg \in A$, implies $abG \subseteq A$ or $ag \in A$ or $bg \in A$.

Therefore, $h(abG) \subseteq h(A)$ or $h(ag) \in h(A)$ or $h(bg) \in h(A)$. As h is an N-homomorphism,

we have $abh(G) \subseteq h(A)$ or $ah(g) \in h(A)$ or $bh(g) \in h(A)$.

Therefore, $abG' \in h(A)$ or $ag' \in h(A)$ or $bg' \in h(A)$.

That is, $ab \in (h(A): G^1)$ or $ag' \in h(A)$ or $bg' \in h(A)$.

This shows that $h(A)$ is a c-2-absorbing ideal of G^1.

Theorem 3.12 : *Let $h: G \to G^1$ be an N-epimorphism. If A^1 is a c-2-absorbing ideal of G^1, then $h^{-1}(A^1)$ is a c-2-absorbing ideal of G.*

Proof : Suppose A^1 is c-2-absorbing ideal of G'. Let $a, b \in N$, $g \in G$ be such that $abg \in h^{-1}(A^1)$. Since h is an N-homomorphism, it follows that $abh(g) = h(abg) \in A^1$. Again, since A^1 is c-2-absorbing, it follows that,

$$ag' \in A^1 \text{ or } bg' \in A^1 \text{ or } ab \in (A^1: G^1).$$

Case (i) : If $ab \in (A^1: G^1)$, then $abG^1 \subseteq A^1$ and $h^{-1}(abG^1) \subseteq h^{-1}(A^1)$. Since h is an N-epimorphism, $abh^{-1}(G^1) = abG \subseteq h^{-1}(A^1)$, shows that $ab \in (h^{-1}(A^1): G)$.

Case (ii) : If $ag' \in A^1$, then $ah^{-1}(g') \in h^{-1}(A^1)$, shows that $ag \in h^{-1}(A^1)$. Similarly, $bg' \in A^1$, then $bg \in h^{-1}(A^1)$. Thus, $h^{-1}(A^1)$ is c-2-absorbing.

Lemma 3.13 : *(Juglal et.al. [16]) For any ideal A of G, (A: G) is an ideal of N.*

For completeness, we provide the proof of the following lemma.

Lemma 3.14 : *If A is a c-prime ideal of G, then (A: G) is a c-prime ideal in N.*

Proof : Since A $\neq G$, $1 \notin (A: G)$, and $(A: G)$ is proper. Suppose that $ab \in (A: G)$ and $b \notin (A: G)$. So $bG \not\subset A$, implies that, there exists $g \in G$ with $bg \notin A$. Now a$(bg) = (ab)g \in A$. Since A is c-prime in G, it follows that $aG \subseteq A$. This shows that $a \in (A: G)$. Therefore, $(A: G)$ is a c-prime ideal of N.

Theorem 3.15 : *If A is a c-prime ideal of G, then A is c-2-absorbing in G.*

Proof : Let A be a c-prime ideal of G, and let $a, b \in N$, $g \in G$ with $abg \in A$. Since A is c-prime, it follows that $abG \subseteq A$ or $g \in A \Rightarrow ab \in (A: G)$ or $g \in A$. Now $ab \in (A: G)$ and since $(A: G)$ is a c-prime ideal in N, it follows that $a \in (A: G)$ or $b \in (A: G)$, and get $ag \in A$ or $bg \in A$. Therefore, A is a c-2-absorbing ideal in G.

An N-group G is monogenic (Pilz [25]), if there corresponds an element $g \in G$ such that $Ng = G$, and G is locally monogenic (Wen Ke Fong and Meyer [17]), if for every $S \subseteq G$, where S is finite, there corresponds an element $a \in G$ such that $S \subseteq Na$.

It is obvious that every locally monogenic N-group is monogenic.

Note 3.16 : Let A be an ideal of G. Then $(A: G) \subseteq (A: g)$ for all $g \in G \backslash A$.

Lemma 3.17 : *Let G be monogenic by $g \in G$. Then $(A: g) = (A: G)$ for every ideal A of G.*

Proof : Since G is monogenic by g, we have $Ng = G$. In this case, $G = \langle g \rangle$. Take $x \in (A: g) \Rightarrow xg \in A \Rightarrow x \langle g \rangle \subseteq A$. This means $xG \subseteq A$ and so $x \in (A: G)$. Therefore, $(A: g) \subseteq (A: G)$.

Conversely, let $x \in (A: G)$. Then $xG \subseteq A$. In particular, $xg \in A$ and so $x \in (A: g)$.

Therefore $(A: G) \subseteq (A: g)$, and hence $(A: g) = (A: G)$.

Theorem 3.18 : *Let G be monogenic and A be an ideal of G. If (A: G) is a c-2-absorbing ideal in N then A is a c-2-absorbing ideal of G.*

Proof : Let $(A: G)$ be a c-2-absorbing ideal in N. To prove A is a c-2-absorbing ideal of G, take $a, b \in N$ and $x \in G$ with $abx \in$ A. Since G is monogenic, we have $x = cg$ for some $c \in N$. Now $abcg = abx \in A$, implies $abc \in (A: g)$, and by lemma 3.17, we get $abc \in (A: G)$. Now $abc \in (A: G)$, and since $(A: G)$ is a c-2-absorbing ideal of N, we have $ab \in (A: G)$ or $ac \in (A: G)$ or $bc \in (A: G)$. That is, $ab \in (A: G)$ or $acg \in A$ or $bcg \in A$ and hence $ab \in (A: G)$ or $ax \in A$ or $bx \in A$, shows that A is c-2-absorbing.

Definition 3.19 : (Meldrum [22]) We say that N distributes over G if $d(g_1 + g_2) = dg_1 + dg_2$ for all $d \in N$, $g_1, g_2 \in G$.

Lemma 3.20 : *If G is locally monogenic and N is distributive, then N distributes over G.*

Proof : Suppose G is locally monogenic and N is distributive. Take $n \in N$ and $g_1, g_2 \in G$. Since G is locally monogenic and $\{g_1, g_2\} \subseteq G$, we have $G \subseteq Ng$, for some $g \in G$. So there exists $n_1, n_2 \in N$ such that $g_1 = n_1 g$ and $g_2 = n_2 g$. Now, $n(g_1 + g_2) = n(n_1 g + n_2 g) = n((n_1 + n_2))g = (nn_1 + nn_2)g = nn_1 g + nn_2 g = ng_1 + ng_2$. Therefore, N distributes over G.

We use lemma 3.20, but also provide explicit verification in the proof of theorem 3.21.

Theorem 3.21 : *Let A be a c-2-absorbing ideal of G, where G is locally monogenic over a distributive near-ring N. Then (A: G) is a c-2-absorbing ideal of N.*

Proof : Suppose that A is c-2-absorbing in G. To show $(A: G)$ is c-2-absorbing in N, let $a, b, c \in N$ and $abc \in (A: G)$. Assume that $ac \notin (A: G)$, $bc \notin (A: G)$. In this case, we show that $ab \in (A: G)$. Since $acG \not\subset A$ and $bcG \not\subset A$, we have $acg_1 \notin A$ and $bcg_2 \notin (A: G)$ for some $g_1, g_2 \in G$. Further, since $abcg \in A$ for all $g \in G$, $abcg_1, abcg_2 \in A$, and since A is an additive subgroup of G, it follows that $abcg_1 + abcg_2 \in A$. Since G is locally monogenic and $\{cg_1, cg_2\} \subseteq G$, it follows that $cg_1 = n_1g_k$, $cg_2 = n_2g_k$ for some $n_1, n_2 \in N$, $g_k \in G$.

Now $ab(cg_1 + cg_2) = ab(n_1g_k + n_2g_k) = ab((n_1 + n_2)g_k) = (ab(n_1 + n_2))g_k$
$= (abn_1 + abn_2)g_k$ (as ab is distributive element in N)
$= abn_1g_k + abn_2g_k = abcg_1 + abcg_2$

Therefore, $ab(cg_1 + cg_2) \in A$. Since A is c-2-absorbing, it follows that $a(cg_1 + cg_2) \in A$ or $b(cg_1 + cg_2) \in A$ or $ab \in (A: G)$. If $a(cg_1 + cg_2) \in A$, then by lemma 3.20, $acg_1 + acg_2 \in A$. Now if $acg_2 \in A$, then $acg_1 = acg_1 + acg_2\text{-}acg_2 \in A$, a contradiction.

Therefore, $a(cg_2) \notin A$, also $b(cg_2) \notin A$ but $ab(cg_2) \in A$ (here, $cg_2 \in G$). Since A is c-2-absorbing, we have $ab \in (A: G)$. In a similar way if $b(cg_1 + cg_2) \in A$, then $bcg_1 + bcg_2 \in A$. If $bcg_1 \in A$, then since $bcg_1 + bcg_2 = i_1$ (say $i_1 \in A$), we get $bcg_2 = i_1\text{-}bcg_1 \in A$, a contradiction. Therefore, $bcg_1 \notin A$, also $acg_1 \notin A$ but $abcg_1 \in A$. Since A is c-2-absorbing, it follows that $ab \in (A: G)$.

Definition 3.22 : (Van der Walt [28]): An N-group G is called connected if for any $g_1, g_2 \in G$, there exists $g \in G$ and $u, v \in N$ such that $g_1 = ug$ and $g_2 = vg$.

Corollary 3.23 : *Let G be a connected N-group over a distributive near-ring N. If A is c-2-absorbing in G, then (A: G) is c-2-absorbing in N.*

Proof : Since G is connected, G is monogenic, and hence the proof follows from theorem 3.21

Proposition 3.24 : Let G be locally monogenic over a distributive near-ring N. If an ideal A of G is c-2-absorbing in G, having IFP, then $(A: g)$ is c-2-absorbing in N for all $g \in G \backslash A$.

Proof : If $1 \in (A: g)$, then $g = 1.\, g \in A$, a contradiction. Therefore, $(A: g) \neq N$ and hence $(A: g)$ is proper. Since $0 = 0g \in A$, and so $0 \in (A: g)$, hence $(A: g) \neq \phi$. Now by proposition 1.42 of Pilz [25], $(A: g)$ is an ideal of N.

To show that $(A: g)$ is c-2-absorbing, let $a, b, c \in N$ and $abc \in (A: g)$. We need to show $ab \in (A: g)$ or $bc \in (A: g)$ or $ac \in (A: g)$. Since $abc \in (A: g)$, we have a$(bc)g = abcg \in A$. Since A is c-2-absorbing in G, it follows that $ag \in A$ or $bcg \in A$ or $abc \in (A\colon G)$.

Case (i) : If $ag \in A$, then since A has IFP, we get $ang \in A$ for all $n \in$ N. In particular, $abg \in A$ and so $ab \in (A\colon g)$.

Case (ii) : If $bcg \in A$, then clearly, $bc \in \in (A\colon \mathrm{g})$. If $abc \in (A\colon G)$, then the proof follows from the theorem 3.21.

Theorem 3.25 : *Let P, Q be ideals of G with $Q \subseteq P$. Then P is a c-2-absorbing ideal of G if and only if P/Q is a c-2-absorbing ideal of G/Q.*

Proof : Assume that P is a c-2-absorbing ideal of G. Then, G/P is an N-group by natural way $n(g + Q) = ng + Q$ for all $n \in N$, $g \in$ G and P/Q is an ideal of G/Q. Now $abg + Q \in P/Q$, for some $i \in P$, implies that $abg + Q = i + Q$, for some $i \in P$, and so $abg\text{-}i \in Q \subseteq P$.

Since P is an ideal of G, we have $abg = abg\text{-}i + i \in P$. Now since P is c-2-absorbing, $ab \in (P\colon \mathrm{G})$ or $ag \in P$ or $bg \in P$, shows that $abG \in P$ or $ag \in P$ or $bg \in P$. That is, $ab(G/P) \subseteq P/Q$ or a$(g + Q) \in P/Q$ or $b(g + Q) \in P/Q$. Hence, P/Q is *a* c-2-absorbing ideal of G/Q.

Conversely, suppose that P/Q is *a* c-2-absorbing ideal of G/Q. Since $P/Q \neq G/Q$, we have P is proper. Let $a, b \in N$ and $g \in G$ be such that $abg \in P$. Then $ab(g + P) = abg + Q \in P/Q$. Since P/Q is c-2-absorbing, we get a$(g + P) \in P/Q$ or $b(g + P) \in P/Q$ or $ab(G/Q) \subseteq P/Q$. This shows that, $ag \in P$ or $bg \in P$ or $abG \subseteq P$, hence P is c-2-absorbing in G.

4. *c*-2-absorbing ideals of module over zero symmetric near-ring

In this section, we consider an N-group G where N is a zero symmetric near-ring. We give examples of c-2-absorbing ideals and obtain some properties.

Example 4.1 : Here $N = \{0, a, b, c\}$. We define binary operations + and $\cdot$ as respectively in table 5 and table 6 below.

Table 5

+	0	*a*	*b*	*c*
0	0	*a*	*b*	*c*
a	*a*	0	*c*	*b*
b	*b*	*c*	0	*a*
c	*c*	*b*	*a*	0

Table 6

·	0	*a*	*b*	*c*
0	0	0	0	0
a	0	0	*a*	0
b	0	0	*b*	0
c	0	0	*c*	0

Here, $A = \{0, b\}$ is a c-prime. Further, it can be verified that A is c-2-absorbing.

Example 4.2 : Consider the near-ring N, given in E-3, page 408 of Pilz [25], where $N = \{0, a, b, c\}$ with operations + and $\cdot$ defined in table 7 and table 8 as follows:

Table 7

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 8

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	0	c	c

Observe that N is zero symmetric. Let $A = \{0\}$. Then $ba = 0 \in A$ but $b \notin A$ and $a \notin A$. Therefore, $A = \{0\}$ is not c-prime. However, $A = \{0\}$ is c-2-absorbing. Thus, N is c-2-prime.

Example 4.3 : Let $N = (Z_8, +_8, \cdot_8)$. Take $G = D_8 = \langle\{\sigma, s \mid \sigma^4 = 1, s^2 = 1, \sigma s = s\sigma^{-1}\}\rangle$. Define external operation $*$ as in table 9 below.

Table 9

External operation $*$.

$*$	e	σ	σ^2	σ^3	s	sσ	sσ^2	sσ^3
$\bar{0}$	e	e	e	e	e	e	e	e
$\bar{1}$	e	σ	σ^2	σ^3	s	sσ	sσ^2	sσ^3
$\bar{2}$	e	σ^2	e	σ^2	e	e	e	e
$\bar{3}$	e	σ^3	σ^2	σ	s	sσ	sσ^2	sσ^3
$\bar{4}$	e	e	e	e	e	e	e	e
$\bar{5}$	e	σ	σ^2	σ^3	s	sσ	sσ^2	sσ^3
$\bar{6}$	e	σ^2	e	σ^2	e	e	e	e
$\bar{7}$	e	σ^3	σ^2	σ	s	sσ	sσ^2	sσ^3

Then, G is a N-group. Here, $\{e\}$ is a c-2-absorbing ideal, but $\{e\}$ is not a c-prime ideal.

Example 4.4 : Let $G = D_8 = \langle\{\sigma, s \mid \sigma^4 = s^2 = e, \sigma s = s\sigma^{-1}\}\rangle = \{e, \sigma, \sigma^2, \sigma^3, s, s\sigma, s\sigma^2, s\sigma^3\}$, where σ is the rotation in ant-clockwise direction about the origin through $\frac{\pi}{2}$ radians and s is the reflection about the line of symmetry. Take $G = N$ (listed as no. K(139) on p. 418 of Pilz [25]), and has addition and multiplication tables are given in table 10 and table 11 as follows (also refer to Hamsa et.al. [12]).

Table 10

+	e	σ	σ^2	σ^3	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$
e	e	σ	σ^2	σ^3	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$
σ	σ	σ^2	σ^3	e	$s\sigma^3$	s	$s\sigma$	$s\sigma^2$
σ^2	σ^2	σ^3	e	σ	$s\sigma^2$	$s\sigma^3$	s	$s\sigma$
σ^3	σ^3	e	σ	σ^2	$s\sigma$	$s\sigma^2$	$s\sigma^3$	s
s	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$	e	σ	σ^2	σ^3
$s\sigma$	$s\sigma$	$s\sigma^2$	$s\sigma^3$	s	σ^3	e	σ	σ^2
$s\sigma^2$	$s\sigma^2$	$s\sigma^3$	s	$s\sigma$	σ^2	σ^3	e	σ
$s\sigma^3$	$s\sigma^3$	s	$s\sigma$	$s\sigma^2$	σ	σ^2	σ^3	e

Table 11

*	e	σ	σ^2	σ^3	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$
e	e	e	e	e	e	e	e	e
σ	e	σ	σ^2	σ^3	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$
σ^2	e	σ^2	e	σ^2	e	e	e	e
σ^3	e	σ^3	σ^2	e	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$
s	e	s	e	s	$s\sigma^2$	e	$s\sigma^2$	e
$s\sigma$	e	$s\sigma$	σ^2	$s\sigma^3$	σ^2	$s\sigma$	e	$s\sigma^3$
$s\sigma^2$	e	$s\sigma^2$	e	$s\sigma^2$	$s\sigma^2$	e	$s\sigma^2$	e
$s\sigma^3$	e	$s\sigma^3$	σ^2	$s\sigma$	σ^2	$s\sigma$	e	$s\sigma^3$

Then G is an N-group, where N is non-abelian. $I = \{e, \sigma^2\}$ is c-2-absorbing but not c-prime, since $s\sigma * s = \sigma^2 \in I$, but $s \notin I$ and $s\sigma * G \not\subset I$.

We provide the notion of symmetric ideal which is analogous to the notion defined by Lambek [20].

Definition 4.5 : An ideal A of G is said to be symmetric if for $a, b \in N$, $g \in G$, $abg \in A$ implies $bag \in A$.

It is evident that an ideal A of N-group N, where N is commutative, is a symmetric ideal.

Theorem 4.6 : *Let G be an N-group, where N is zero-symmetric. If $A = A_1 \cap A_2$ is symmetric where $A_{i,}$ (i = 1, 2) are c-prime ideals of G, then A is a c-2-absorbing ideal of G.*

Proof : Let A_1 and A_2 be c-prime ideals of G. To show $A = A_1 \cap A_2$ is a c-2-absorbing ideal of G, let $a, b \in N$, $g \in G$ with $abg \in A$. Since $A = A_1 \cap A_2$ and $abg \in A$, we have $abg \in A_1$ and $abg \in A_{2.}$ Now $(ab)g \in A_1$ and A_1 is c-prime, we have $abG \subseteq A_1$ or $g \in A_1 \Rightarrow ab \in (A_1: G)$ or $g \in A_1 \Rightarrow a \in (A_1: G)$ or $b \in (A_1: G)$ or $g \in A_1 \Rightarrow a \in (A_1: G)$ or $b \in (A_{1:}\ G)$ or $g \in A_1$, by lemma 3.14.

In a similar argument, we get $abg \in A_2 \Rightarrow a \in (A_2: G)$ or $b \in (A_2: G)$ or $g \in A_{2.}$

Case (i) : If a $\in (A_1: G)$ and $a \in (A_{2:}\ G)$, then $a \in ((A_1: G) \cap (A_{2:}\ G)) = (A_1 \cap A_2: G)$, implies that $ag \in A_1 \cap A_2$ for all $g \in G$. Now $bag = b(0 + ag)\text{-}b0 \in A_1 \cap A_2$. Since A is symmetric and $bag \in A_1 \cap A_2$, it follows that $abg \in A_1 \cap A_2$ for all $g \in G$, which means that $ab \in (A_1 \cap A_2: G)$. Therefore, in this case we have obtained that $A_1 \cap A_2$ is c-2-absorbing.

Case (ii) : Let $a \in (A_1: G)$ and $b \in (A_{2:}\ G)$. Then by lemma 3.13, we have $ab \in (A_1 \cap A_2: G)$. This implies that $abG \subseteq A_1 \cap A_2$. Hence, $A_1 \cap A_2$ is c-2-absorbing ideal of G.

Case (iii) : Let $a \in (A_1: G)$ and $g \in A_{2.}$ Then $ag \in A_1$ for all $g \in G$. Since $g \in A_2$ and A_2 is an ideal of G, it follows that $ag = a(0_G + g)\text{-}a0_G \in A_2$. Therefore, $ag \in A_1 \cap A_{2.}$ Hence, $A_1 \cap A_2$ is a c-2-absorbing ideal of G.

Theorem 4.7 : *Let A be a c-2-absorbing ideal of G. Then*

(i) *(A: G) is a c-prime ideal of N implies (A: g) is a c-prime ideal of N for all g $\in G \backslash A$.*

(ii) *If (A: g) is a c-prime ideal of N for all g $\in G \backslash A$, then (A: G) is c-prime if N is zero-symmetric and distributive over G.*

Proof : (i) Suppose $(A: G)$ is a c-prime ideal of N. Let $a, b \in N$, $g \in G \backslash A$ and $ab \in (A: g)$. Then $abg \in A$. Since A is c-2-absorbing, $ag \in A$ or $bg \in A$ or $ab \in (A: G)$.

Case (i) : If $ag \in A$, then $a \in (A: g)$.

Case (ii) : If $bg \in A$, then $b \in (A: g)$.

Case (iii) : Let $ab \in (A: G)$. Then since $(A: G)$ is c-prime, we have $a \in (A: G)$ or $b \in (A: G)$. Therefore, by note 3.16, $a \in (A: g)$ or $b \in (A: g)$.

Hence, $(A: g)$ is a c-prime ideal of N.

(ii) Suppose $(A: g)$ is a c-prime ideal of N for all $g \in G \setminus A$.

We prove that $(A: G)$ is a c-prime ideal of N. Clearly $(A: G)$ is an ideal of N. In a contrary way suppose that $ab \in (A: G)$ for some $a, b \in N$ and $a \notin (A: G)$, $b \notin (A: G)$. Then there exist $g, g' \in G$ such that $ag \notin A$, $bg' \notin A$. Since $abg \in A$ and $abg' \in A$, we have $ab \in (A: g)$ and $ab \in (A: g')$. Now by hypothesis $b \in (A: g)$ and $a \in (A: g')$ implies that $bg \in A$ and $ag' \in A$.

If $g + g' \in A$, then $a(g + g') = a(0 + (g + g'))\text{-}a0 \in A$. Since N distributes over G an $ag' \in A$, it follows that $ag = ag + ag'\text{-}ag' \in A$, a contradiction. Therefore, $g + g' \notin A$. In a similar argument, since N distributes over G, we have $ab(g + g') \in A$. By hypothesis, $(A: g + g')$ is a c-prime ideal of N and so $a(g + g') \in A$ or $b(i + g') \in A$, a contradiction. Hence, $(A: G)$ is a c-prime ideal of N.

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References

[1] A. Badawi. On 2-Absorbing ideals of commutative rings. *Bull. Austral. Math. Soc.*, 75(3):417-429, 2007. DOI:10.1017/S0004972700039344.

[2] S. Bhavanari, and G. Koteswara Rao. On a class of modules and N-groups. *J. Indian Math Soc.*, 59:39-44, 1993.

[3] S. Bhavanari, Lokeswara Rao and S.P. Kuncham. A note on Primeness in Near-rings and Matrix Near-rings. *Indian J. Pure and Appl. Math.,* 27: 227-234 1996.

[4] S. Bhavanari, S.P. Kuncham, and TVP. Kumar. On IFP N-Groups and Fuzzy IFP Ideals. *Indian Journal of Mathematics*, 46(1): 11-19, 2004.

[5] S. Bhavanari, and S.P. Kuncham. Near-rings, Fuzzy ideals and Graph theory, *CRC press* (Taylor and Francis, U.K, U.S.A), 2013. ISBN 9780367380045.

[6] A.Y. Darani and F. Soheilnia. 2-absorbing and weakly 2-absorbing submodules. *Thai J. Math.*, 9(3): 577-584, 2011.

[7] A.D. Diallo and P.C. Diop. Modules whose t-closed submodules are their homomorphic images. *Journal of Discrete Mathematical Sciences and Cryptography*, 23(6): 1297-314, 2020. DOI:10.1080/09720529.2020 .1815339

[8] N.J. Groenewald and D. Ssevviiri. Kothe's upper nil radical for modules. *Acta Math Hungary*, 138(4): 295-306, 2013.

[9] N.J. Groenewald and D. Ssevviiri. Completely prime submodules. *International Electronic Journal of Algebra*, 13(1): 1-14, 2013.

[10] N.J. Groenewald and Bac T. Ngnyen. On 2-absorbing modules over Non-commutative rings. *International Electronic Journal of Algebra*, 25: 212-223, 2019. DOI:10.24330/ieja.504155.

[11] N.J. Groenewald. Weakly prime and weakly completely prime ideals of non-commutative rings. *International Electronic Journal of Algebra*, 28: 43-60, 2020. DOI:10.24330/ieja.768127.

[12] N. Hamsa, S.P. Kuncham, and B.S. Kedukodi, ΘΓ-Ngroups, *Matematicki Vesnik*, 70(1): 64-78, 2018.

[13] W.H. Hanoon, S.M. Aljassa, and M.S. Fiadh. Nearly 2-absorbing and weakly nearly 2-absorbing fuzzy submodules. *Journal of Discrete Mathematical Sciences and Cryptography*, 23(5): 1025-1037, 2020. DOI:10 .1080/09720529.2020.1714885.

[14] M.M. Hasnain and M.R. Khan. On prime near-rings with derivations. *Journal of Discrete Mathematical Sciences and Cryptography*, 20(3): 695-702, 2017. DOI:10.1080/09720529.2016.1187956.

[15] W.M.L. Holcombe. Primitive near-rings. *Diss. university of Leeds*, 1970.

[16] S. Juglal, N.J. Groenewald, and K.S.E Lee. Different prime R-ideals. *Algebra Colloquium*, 17(1): 887-904, 2010. DOI:10.1142/ S1005386710000830

[17] W.F. Ke and J.H. Meyer. Matrix near-rings and 0-primitivity. *Monatsh Math.*, 165: 353-363, 2012. DOI:10.1007/s00605-010-0267-z.

[18] B.S. Kedukodi, S. Bhavanari, and S.P. Kuncham. Equiprime, 3-prime and c-prime fuzzy ideals of near-rings. *Soft Computing*, 13(10): 933-944, 2008. DOI:10.1007/s00500-008-0369-x.

[19] B.S. Kedukodi, S. Bhavanari, and S.P. Kuncham. c-prime fuzzy ideals of near-rings. *Soochow J. Math.*, 33(4): 891-901, 2007.

[20] J. Lambek. Lectures on Rings and Modules, *American Mathematical Soc.*, (283), 2009.

[21] U. Leerawat, M. Sirivoravit, and K. Daowsud. On near generalized rings. *Journal of Discrete Mathematical Sciences and Cryptography*, 23: 1-5, 2020. DOI:10.1080/09720529.2020.1732085

[22] J.D.P. Meldrum. Near-rings and their links with groups. *Pitman Adv. publications programs*, 1985.

[23] Y.S. Pawar and P.V. Patil. Semiprime elements and relative annihilators in multiplicative lattices. *Journal of Discrete Mathematical Sciences and Cryptography*, 12(2): 193-204, 2009. DOI:10.1080/09720529.2009.10698230

[24] Sh. Payrovi and S. Babaei. On 2-absorbing submodules. *Algebra Colloquium*, 19(1), 913-920, 2012. DOI:10.1142/S1005386712000776.

[25] G. Pilz. Near-rings. *North Holland*, 1983.

[26] D. Ramakotaiah and G.K. Rao. IFP near-rings. *Journal of Austral Math Soc.*, 27(3): 365-370, 1979. DOI:10.1017/S1446788700012477.

[27] P. Sutida and P. Sajee. *Proc. of the 22nd annual meeting in mathematics* (AMM 2017), Chiang Mai University, Thailand, 1-10, 2017.

[28] A.P. Van Der Walt. Primitivity in matrix near-rings. *Quaestiones Mathematicae*, 9: 459-469, 1986. DOI:10.1080/16073606.1986.9632127.

[29] A.A. Zyarah and N.S. Al-Mothafar. On primary radical R-submodules of R-modules. *Journal of Discrete Mathematical Sciences and Cryptography*, 23(5): 1001-1007, 2020. DOI:10.1080/09720529.2019.1700918.